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A COALITIONAL EXTENSION OF THE ORDINAL SHAPLEY-SHUBIK VALUE

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ABSTRACT

In cooperative game theory it is known that two-person bargaining problems have no relevant ordinal solution. For three-player bargaining problems, Shapley and Shubik propose an ordinal rule. However, this rule does not take into account the worth of proper subcoalitions of size 2. In this paper, we fill this gap by proposing a generalization of the Shapley-Shubik rule for Non transferable utility games. The resulting solutions, when applied to transferable utility games, always belong to the core, which makes it a relevant alternative to other core-selectors such as the nucleolus. We also apply the new solution to a practical case related to mining and natural resources management.

Keywords: NTU-games, Ordinal Shapley-Shubik value, mining, natural resources management.

1. INTRODUCTION

Ordinal bargaining have been a classical research topic in economics and has also recently become an interesting issue in computer science (Aziz et al., 2016; Zhan et al., 2018; Erlich et al., 2018; Gafni et al., 2021; Hazon et al., 2024). Assume there are gains of cooperation when various agents form coalitions but there also conflicts on which coalition should be formed and how the gains should be shared. Hence, the distribution of gains involves bargaining not only between individual agents but also among coalitions. When each agent anticipates its opportunity cost when actually participating in each coalition, its foregone reward may take into account this opportunity cost.

A key aspect is the preference of each agent, usually represented by a utility function. A utility function assigns for each agent a numerical value to each possible outcome, so that a higher utility reflects a preference relation for the agent. However, any given preference can be represented by many possible utility functions.

Hence, a very desirable property for a value is ordinal invariance. In cooperative games, a value satisfies ordinal invariance when it remains unaffected by order-preserving transformations of the agents' utilities. Shapley (1969) demonstrates that, among efficient values, only the dictatorial ones, i.e., those that give a particular agent all the bargaining power, maintain ordinal invariance for two agents.

Take for example the symmetric case in which there are two players, say player 1 and player 2, who have to agree on how to divide one unit of some desirable asset. Assuming selfishness, the utilities can be represented by $u_1(x, y) = x$ and $u_2(x, y) = y$, where (x, y) means that player 1 receives x and player 2 receives y. In this example, the Pareto frontier of the utility that players can achieve by themselves when agreeing to cooperate lies on the set $D = \{(x, 1-x) : x \in [0, 1]\}$. Assume that an optimal solution depends only on D (the utility space) and specifies that the share of the dollar should assign utilities $(\alpha, 1-\alpha) \in D$ for some $\alpha \in [0, 1]$.

Assume now we represent player 1's preferences by the (equivalent) utility function $u_1'(x, y) = x^3$. Hence, the utility space turns into $D' = \{(x^3, 1 - x) : x \in [0, 1]\}$. The previous optimal solution, if ordinal, should then assign utilities $(\alpha^3, 1 - \alpha) \in D'$.

Assume we also change player 2's utility to the (equivalent) function $u_2''(x, y) = 1 + (y - 1)^3$. Hence, the utility space turns back into $D'' = \{(x^3, 1 - x^3) : x \in [0, 1]\} = D$. The optimal and ordinal solution should then, on one hand, assign utilities $(\alpha^3, 1 - \alpha^3) \in D$ (because ordinality), and, on the other hand, $(\alpha, 1-\alpha) \in D$ (because the only relevant information is given by $D'' = D$). Thus, $\alpha = \alpha^3$ and $1-\alpha = 1-\alpha^3$, which, together with $\alpha \in [0, 1]$, is only possible when $\alpha = 0$ (giving all the bargaining power to player 2) or $\alpha = 1$ (giving all the bargaining power to player 1). Symmetry is thus incompatible with optimality and ordinality.

However, this example does not extend to scenarios involving more than two agents. Shubik (1982) initially presents a rule that is efficient, symmetric, and maintains ordinal invariance for three agents. Although the origin of this rule is not explicitly mentioned in Shubik (1982), Pérez-Castrillo and Wettstein (2006)[p. 297] attribute it toShapley (1969). Moreover, Roth (1979)[p. 72-73] discusses a three-agent ordinal bargaining rule attributed to Shapley and Shubik in a 1974 working paper. Subsequently, in line with the works of Kıbrıs (2004b,a), we refer to it as the Shapley-Shubik rule.

Kıbrıs (2004a) delineates a category of three-agent problems that encompass all bargaining problems. Within this category, the ordinal Shapley-Shubik rule aligns with both the Egalitarian rule (Kalai, 1977) and the Kalai-Smorodinsky rule (Kalai and Smorodin- sky, 1975). Furthermore, it stands as the sole symmetric member among a set of ordinal monotone path rules. Kıbrıs (2004b) further demonstrates the deep connection between the ordinal Shapley-Shubik rule and a solution set defined by Bennett (1997) for multilateral bargaining problems. Additionally, Kıbrıs (2012) outlines the characterization of the ordinal Shapley-Shubik rule utilizing a less stringent version of the Independence of Irrelevant Alternatives (Nash, 1950). From a non-cooperative point of view, Vidal-Puga (2015) presents a game that yields the ordinal Shapley-Shubik rule in subgame perfect equilibria.

Conversely, Samet and Safra (2005) expand the ordinal Shapley-Shubik rule to accommodate scenarios with more than three agents, employing methodologies akin to those in the work of O'Neill et al. (2004). Additionally, Safra and Samet (2004) introduce another set of ordinal solutions.

Adopting an alternative methodology,Pérez-Castrillo and Wettstein (2006)as well as Zhang and Zhang (2008) leverage the inherent physical framework that produces the frontier of utility possibilities. This approach enables Pérez-Castrillo and Wettstein (2006) to establish an ordinal expansion for the Shapley value applicable to any number of agents. Calvo and Peters (2005) present a blended method, examining scenarios involving both ordinal and cardinal agents. Unlike the ordinal Shapley-Shubik value and the extension by Samet and Safra (2005), Pérez-Castrillo and Wettstein (2006) and Calvo and Peters (2005) permit partial agreements within subcoalitions, thus referring to non-transferable utility games.

In this paper, we present a generalization of the ordinal Shapley-Shubik rule to the case of non-transferable utility games. As opposed to Pérez-Castrillo and Wettstein (2006) and Zhang and Zhang (2008), the problem is formulated in the utility space. Hence, we keep the classical assumption that only utilities that coalitions can achieve by themselves are relevant in the game. Moreover, as opposed to Calvo and Peters (2005), we only consider ordinal players. We also show that our generalization satisfies core-selection when restricted to transfer-utility games, which makes it a reasonable alternative to other well-known values such as the nucleolus or the Shapley value.

The paper is organized as follows. In Section 2, we present the model and describe some previous results. In Section 3, we define a generalization of the ordinal Shapley-Shubik rule and prove some of its properties. Proofs can be required to the authors. In Section 4, we implement this theoretical framework in a practical context by investigating its applicability within the specific setting of Winikunka Mountain, also recognized as the Seven Color Mountain, located in Peru.

2. NOTATION AND PREVIOUS RESULTS

Let $N = \{1, 2, 3\}$ be the fixed set of agents. For each nonempty $S \subseteq N$ and $x, y \in \mathbb{R}^S$, $x \le y$ means $x_i \le y_i$ for all $i \in S$, $x \ll y$ means $x_i < y_i$ for all $i \in S$, and $x < y$ means $x \le y$ and $x \ne y$.

Given $S \subset N$ and $x \in \mathbb{R}^N$, x_S is the restriction of x in S, i.e., $x_S \in \mathbb{R}^N$ is defined as $(x_S)_i = x_i$ for all $i \in S$.

Given $S \subseteq N$ and a surface $A \subset \mathbb{R}^S$, a point $x \in A$ is *Pareto optimal* in A if there is no $y \in A$ such that $x < y$. Let $PO\,(A)$ denote the set of Pareto optimal points in A. A point $x \in A$ is *weakly Pareto optimal* in A if there is no $y \in A$ such that $x \ll y$. Let $WPO\,(A)$ denote the set of weakly Pareto optimal points in A.

Definition 1 *A (three-player)* Non-transferable utility (NTU) game *is a characteristic function $V : S \in 2^N \to V(S) \in \mathbb{R}^S$ where $V(S)$ is the set of feasible utility assignments for each coalition S, so that each $V(S)$ is non-empty, closed, comprehensive and bounded from above.*

It is clear from the definition of a NTU game V that there exists some $d \in \mathbb{R}^N$ such that $V(\{i\}) = (-\infty, d_i]$ for all $i \in N$.

An NTU game V is *strongly comprehensive* if, for all $S \subseteq N$, $WPO\,(V(S)) = PO\,(V(S))$ and for each $x \in V(S)$, $y \le x$ implies $y \in V(S)$. Let $\mathcal{SC}$ denote the set of all strongly comprehensive NTU games. When $v \in \mathcal{SC}$, we write $\partial V(S)$ instead of $P(V(S))$ or $WP(V(S))$.

Given a class of NTU games $\mathcal{G}$, a *value* ϕ on $\mathcal{G}$ is a function that assigns to each NTU game $V \in \mathcal{G}$ a subset $\phi(V) \subset \mathbb{R}^N$ of payoff allocations that represent the utility assigned to each agent in the game.

Three known values for NTU games are the Harsanyi value (Harsanyi, 1963; Imai, 1983; Hart, 1985; de Clippel et al., 2002), the Shapley NTU value (Shapley, 1969; Au- mann, 1985), and the consistent value (Maschler and Owen, 1992; de Clippel et al., 2002; Hart, 1994, 2005). All three of them coincide with the Shapley value (Shapley, 1953) for TU games and with the Nash rule for bargaining problems:

Harsanyi value A point $Ha(V) \in PO(V(N))$ is a *Harsanyi value* of V if there exists $\lambda \in \mathbb{R}^N_+$ such that, for each coalition $S \subseteq N$, there exist $H^{\lambda,S} \in PO(V(S))$ and $\xi^{\lambda,S} \in \mathbb{R}$ such that $Ha(V) = H^{\lambda,N}$ and

$$\lambda_i H_i^{\lambda,S} = \sum_{T \subseteq S : i \in T} \xi^{\lambda,T}$$

for all $i \in S \subseteq N$.

Shapley NTU value A point $Sh(V) \in PO(V(N))$ is a *Shapley NTU value* of V if there exists $\lambda \in \mathbb{R}^N_{++}$ such that $\lambda_i Sh_i(V) = Sh_i(v^\lambda)$ for all $i \in N$, where $Sh(v^\lambda)$ is the Shapley value of the TU game v^λ defined as $v^\lambda(\emptyset) = 0$ and

$$v^\lambda(S) = \max_{x \in V(S)} \sum_{i \in S} \lambda_i x_i$$

for each $S \subseteq N \setminus \{\emptyset\}$ whenever these maxima exist.

Consistent value Let Π^N denote the set of orders of the agents in N. For each $i \in N$ and $\pi \in \Pi^N$, let $P_i^\pi = \{j \in N : \pi(j) < \pi(i)\}$ be the set of predecessor of player i in π. A point $Co(V) \in PO(V(N))$ is a *consistent value* of V if there exist $\lambda^S \in \mathbb{R}^S_{++}$ for each $S \subseteq N$ such that $v^{\lambda^S}(S)$ is well-defined for each $S \subseteq N$ and

$$Co_i(V) = \frac{1}{|\Pi^N|} \sum_{\pi \in \Pi^N} Co_i^\pi(V)$$

for all $i \in N$, where $Co^\pi(V) \in \mathbb{R}^N$ is defined inductively as

$$Co_i^\pi(V) = \frac{1}{\lambda_i^{P_i^\pi \cup \{i\}}} \left(v^{\lambda^{P_i^\pi \cup \{i\}}} (P_i^\pi \cup \{i\}) - \sum_{j \in P_i^\pi} \lambda_j^{P_i^\pi \cup \{i\}} Co_j^\pi(V) \right)$$

for all $i \in N$.

Desirable properties of a value are efficiency, anonymity, individual rationality, core selection, and ordinal invariance. We describe each of these properties.

Efficiency A value ϕ is *efficient* when $\phi(V) \subseteq PO(V(N))$ for each NTU game V.

Let Π be the set of all permutations of N, with generic element π. Given $\pi \in \Pi^N$, $S \subseteq N$ and $A \subseteq \mathbb{R}^S$, we define $\pi(S) = \{\pi(i) : i \in S\}$ and $\pi(A)$ as

$$\pi(A) = \left\{ \left(x_{\pi(i)}\right)_{i \in S} \in \mathbb{R}^{\pi(S)} : x \in A \right\}.$$

Moreover, πV is the NTU game defined by $\pi V(\pi(S)) = V(S)$ for all $S \subseteq N$.

Anonymity A value ϕ is *anonymous* when it is covariant with respect to any permutation of players, i.e, given $\pi \in \Pi^N$,

$$\phi_{\pi(i)}(\pi V) = \phi_i(V)$$

for all $i \in N$.

An NTU game V is *superadditive* if $V(S) \times V(T) \subseteq V(S \cup T)$ for all $S, T \subset N$ with $S \cap T = \emptyset$.

Individual rationality A value ϕ is *individually rational* if $d \leq x$ for all $x \in \phi(V)$ for each superadditive NTU game V.

The *core* of an NTU game V is defined as the set of efficient payoff allocations that cannot be improved by any coalition, i.e.,

$$Core(V) = \{x \in PO(V(N)) : x_S \notin Int(V(S)) \forall S \subset N\}$$

where

$$Int(V(S)) = V(S) \setminus WPO(V(S))$$

is the interior of $V(S)$, i.e., those points that can be improved by coalition S, in the sense that given $x \in Int(V(S))$, there exist payoffs allocations in $V(S)$ that improve x_i for each $i \in S$.

Core selection A value ϕ is a *core selector* if $\phi(V) \subseteq Core(V)$ whenever $Core(V) \neq \emptyset$.

Given $x \in \mathbb{R}^N$ and $(f_i)_{i \in N}$ a vector of strictly increasing functions $f_i : \mathbb{R} \to \mathbb{R}$, we define

$$f(x) = \left(f_i(x_i)\right)_{i \in N} \in \mathbb{R}^N$$

and, given $X \subset \mathbb{R}^N$, we define

$$f(X) = \{f(x) : x \in X\}.$$

Moreover, fV is the NTU game defined by

$$fV(S) = \left\{ (f_i(x_i))_{i \in S} \in \mathbb{R}^S : x \in V(S) \right\}$$

for all $S \subseteq N$.

Ordinal invariance A value ϕ is *ordinal* if it is not affected by order-preserving changes in utility, i.e., given $(f_i)_{i \in N}$ with $f_i : \mathbb{R} \to \mathbb{R}$ strictly increasing,

$$\phi(fV) = f(\phi(V)).$$

A particular class of NTU games are *Transfer-Utility (TU) games*. Formally, V is a TU game if there exists $v : 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$ such that

$$V(S) = \left\{ x \in \mathbb{R}^S : \sum_{l \in S} x_l \leq v(S) \right\}$$

for all $S \subseteq N$. In that case, we say that V can be represented as a TU game v. It is straightforward to check that any TU game belongs to $\mathcal{SC}$.

An NTU game that can be represented as a TU game v is superadditive if and only if $v(S) + v(T) \leq v(S \cup T)$ for all $S, T \subset N$ with $S \cap T = \emptyset$. Let $\mathcal{STU}$ denote the set of superadditive TU games.

Another particular class of NTU games are *bargaining problems*. These are NTU games in which unanimity is required to reach an agreement, and otherwise an status quo, or default payoff allocation, is implemented.

Formally, V is a bargaining problem if $V(S) = \prod_{i \in S}(-\infty, d_i]$ for all $S \subsetneq N$. Hence, a bargaining problem V is completely determined by the pair $(V(N), d)$.

The ordinal Shapley-Shubik rule applies for bargaining problems in $\mathcal{SC}$. Let $a = (a_1, a_2, a_3) \in \mathbb{R}^N$. Then, there exists a *unique* point $\bar{x} = (\bar{x}_1, \bar{x}_2, \bar{x}_3)$, called *ground point* for a, such that the points $(a_1, \bar{x}_2, \bar{x}_3)$, $(\bar{x}_1, a_2, \bar{x}_3)$, $(\bar{x}_1, \bar{x}_2, a_3)$ are all on $\partial V(N)$ (see Figure 1).

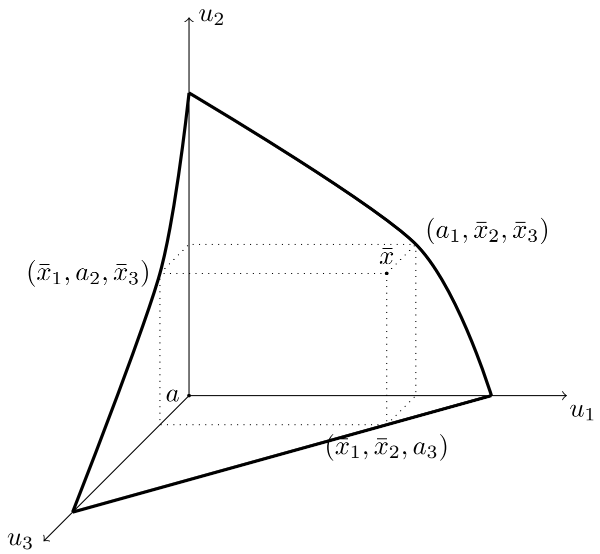

Figure 1: Position of the disagreement points and ground point in $V(N)$ assuming $a = (0, 0, 0)$.

When a belongs to the interior of $V(N)$, its ground point $\bar{x}$ does not belong to $V(N)$, and vice versa. However, the ground point is always closer to the Pareto surface. The *ordinal Shapley-Shubik rule* is defined as the limit point of the sequence of successive ground points that begins with the status quo, i.e.,

$$\varphi(V) = \lim_{t \to \infty} a^t$$

where $a^0 = d$ and a^{t+1} is the (unique) ground point of a^t.

The ordinal Shapley-Shubik rule is efficient, anonymous, individually rational, and ordinal (Safra and Samet, 2004).

Notice that the ordinal Shapley-Shubik rule is defined in bargaining problems, not in general NTU games. This means that 2-player subcoalition worths ($V(S)$ with $|S| = 2$) are not taken into consideration.[1] As far as we know, no ordinal value has been defined for 3-player NTU games yet.

In next subsections, we extend this definition to general NTU games so that the worth of proper subcoalitions are taken into account.

[1]As opposed, 1-player subcoalition worths $V(\{i\})$ are taken into consideration since they are determined by the status quo d.

3. AN ORDINAL VALUE FOR GENERAL NTU GAMES

A trivial extension of the ordinal Shapley-Shubik rule can be achieved by maintaining the same definition without taking into account the role of two-player coalitions, i.e., $V(S)$ with $|S| = 2$. As opposed, we aim to extend this value taking into account these worths. We use the following definition:

Definition 2 *Given a three-player NTU game V, a* baseline *for V is a point $x^V \in \mathbb{R}^N$ such that $x^V_S \in PO(V(S))$ for all $S \subset N$ with $|S| = 2$.*

Existence and unicity of a baseline point is guaranteed in strongly comprehensive NTU games, as next result shows:

Proposition 1 *Given $V \in \mathcal{SC}$, there exists a unique $x^V \in \mathbb{R}^N$ such that $x^V_S \in \partial V(S)$ for all $S \subset N$ with $|S| = 2$.*

An important feature of x^V is that it is ordinal. However, it does not take into account $(V(\{i\}))_{i \in N}$, i.e., the role of one-player coalitions.

Hence, our extension will consider the following intermediate value:

$$x^*_i = \max\left\{d_i, x^V_i\right\} \tag{1}$$

for all $i \in N$.

Remark 1 *If V is a bargaining problem, then $x^* = x^V = d$.*

Recall that $d_i = \max V(\{i\})$ for all $i \in N$.

We then define

$$SS(V) = \lim_{t \to \infty} a^t$$

where $a^0 = x^*$ and a^{t+1} is the (unique) ground point of a^t.

Under Remark 1, SS generalizes the ordinal Shapley-Shubik rule and, for general NTU games, it takes into account the worth of 2-player coalitions. In general, the ordinal Shaple-Shubik value is defined as follows:

Ordinal Shapley-Shubik value Given a three-player NTU game, a point $SS(V) \in \mathbb{R}^N$ is an *ordinal Shapley-Shubik value* if there exists a baseline x^V for V such that $SS(V) = \lim_{t\to\infty} a^t$ where $a^0 = x^*$ defined as $x_i = \max\{d_i, x^V_i\}$ for all $i \in V$, and a^{t+1} is a ground point of a^t for all a^t.

The ordinal Shapley-Shubik value is a singleton when V is strongly comprehensible, as for example in TU games.

Theorem 1 *The ordinal Shapley-Shubik value is efficient, ordinal, and anonymous.*

Next result provides an explicit formula for x^V when V can be represented by a TU game.

Lemma 1 *Assume $N = \{i, j, k\}$ and V can be represented as a TU game v. Then,*

$$x^V_i = \frac{v(\{i,j\}) + v(\{i,k\}) - v(\{j,k\})}{2}. \tag{2}$$

Next result provides an explicit formula for SS in TU games.

Proposition 2 *Assume $N = \{i, j, k\}$ and V can be represented as a TU game v. Then,*

$$x^*_i = \max\left\{v(\{i\}), \frac{v(\{i,j\}) + v(\{i,k\}) - v(\{j,k\})}{2}\right\} \tag{3}$$

and

$$SS_i(V) = x^*_i + \frac{v(N) - \sum_{l \in N} x^*_l}{3} \tag{4}$$

for all $i, j, k \in N$.

Example 1 *Assume V can be represented as a TU game v given by $v(\{i\}) = 0$ for all $i \in N$, $v(\{1,2\}) = 60$, $v(\{1,3\}) = 30$, $v(\{2,3\}) = 24$, and $v(N) = 72$. In this case, $x^V = (33, 27, -3)$. Hence, $x^* = (33, 27, 0)$ and thus $SS(V) = (37, 31, 4)$. As opposed, the Shapley value is $(31, 21, 13)$ and the nucleolus is $(36, 30, 6)$.*

As it can be checked in Example 1, the ordinal Shapley-Shubik value belongs to the core. This result holds in general for TU games with nonempty core, as next Theorem shows.

Theorem 2 *If $V \in \mathcal{STU}$ and $Core(V) \neq \emptyset$, then $SS(V) \in Core(V)$.*

Next example shows that superadditivy in the previous result is essential, as non-superaditive games do not guarantee a core allocation even if such allocations exist.

Example 2 *Assume V can be represented as a TU game v given by $v(\{i\}) = 0$ for all $i \in N$, $v(\{1,2\}) = 60$, $v(\{1,3\}) = 30$, $v(\{2,3\}) = -60$, and $v(N) = 72$. This game is not superadditive because $v(\{2,3\}) < v(\{2\}) + v(\{3\})$. In this case, $x^V = (75, -15, -45)$. Hence, $x^* = (75, 0, 0)$ and thus $SS(V) = (74, -1, -1)$, which does not belong to the core because $SS_2(V) < v(\{2\})$ and $SS_3(V) < v(\{3\})$. Similarly, the Shapley value, $(59, 14, -1)$, does not belong to the core. As opposed, the nucleolus, $(45, 21, 6)$, does belong to the core.*

4. WINIKUNKA MOUNTAIN

In this section, we apply the new defined solution to a real-situation based on the Winikunka mountain in Peru. The Winikunka mountain, also known as Vinicunca or the "siete colores" (seven colors) mountain, situated at 5200 meters above sea level, has become a significant source of income for the surrounding communities, generating jobs and economic opportunities. Nonetheless, concerns over the environmental impact and cultural degradation have intensified, raising questions about the long-term sustainability of the tourism boom. We can recognize three different players in this situation: Local communities, tourist businesses and mining firms.

Assume $N = \{1, 2, 3\}$ where 1 represents the mining industry, 2 the tourism industry, and 3 the local communities. The mining industry is free to choose the level of mining activity in the landscape, normalized to $m \in [0, 1]$, so that $m = 0$ is the minimum (no activity) and $m = 1$ is the maximum activity. The tourism industry can choose a limit on the number of tourists, normalized to $t \in [0, 1]$, where $t = 0$ is the minimum (no tourists allowed) and $t = 1$ is the maximum (no restriction on the number of tourists). Finally, the local communities can organize and hold protests and road blockades, normalized to $p \in [0, 1]$, where $p = 0$ means no protest and $p = 1$ means a total block of roads. Both tourism and mining activity are reduced proportionally to $1 - p$.

We assume that the mining industry prefers a high mining activity (i.e., high m and low p). Analogously, the tourism industry prefers a high t and a low p, and the local communities prefer low mining activity, t as close as possible to some optimal $t^0 \in (0, 1)$, that avoids over-tourism, and also a low p.

Other marginal transfers are possible but negligible with respect to the factor given by m, t, and p. However, we need to take into consideration the possible assets r and s that agent 2 (the tourism industry) can produce to create marginal benefits ϵ and ε to the mining industry and the local communities, respectively. For example, s can be interpreted as free or cheap accommodation for locals in touristic resorts. We also assume $\varepsilon > \epsilon$ as, with equal effort from the tourist industry, the benefit that can be supplied to local communities is bigger than to the mining industry.

The set of alternatives is then given by

$$A = \{(m, t, p, r, s) : m, t, p \in [0, 1], r, s \geq 0\}.$$

Let

$$\tau = \min\{t, (1-p)(1-m)\}$$

be the relative number of tourists, which depends on the quota (t), the mining activity (the more mining activity, the fewer tourists), and the protests (the more demonstrations, the fewer number of tourists).

Player's preferences are represented by the following utility functions:

$$\begin{aligned} u_1 &= (1-p)m + \epsilon r \\ u_2 &= \tau - r - s \\ u_3 &= 1 - \alpha \left|\tau - t^0\right| - \beta(1-p)m - \gamma p + \varepsilon s \end{aligned}$$

where α, β, γ are parameters that determine the relative importance of, respectively, tourism, mining, and protests for locals. In particular

- α represents the (relative) importance of tourism for locals,
- β represents the (relative) importance of the environment for locals, and
- γ represents the (relative) disruption of protests for locals.

We estimate some reasonable values for each of the parameters.

Disruption of protests for locals Question V88 in the 2012 WVS survey in Peru (Inglehart et al., 2014) stated stated "Political action: joining unofficial strikes". Peruvians' normalized opinion (so that 0 is completely for and 1 is completely against) was 0.653 ± 0.03. Hence, we assume $\gamma = 2/3$, inside the confidence interval.

Importance of the environment for locals Question V78 in the 2012 WVS survey in Peru (Inglehart et al., 2014) stated stated "Schwartz: It is important to this person looking after the environment". Peruvians' normalized opinion (so that 0 is not important and 1 is very important) was 0.675 ± 0.03. Hence, we assume $\beta = 2/3$, inside the confidence interval.

Importance of tourism for locals Question V81 in the 2012 WVS survey in Peru (Inglehart et al., 2014) stated "Protecting environment vs. Economic growth". Peruvians' normalized opinion (so that 0 is everybody prefers protecting environment and 1 is everybody prefers economic growth) was 0.314 ± 0.03, with ratio 2.19. Hence, we assume $\beta/\alpha = 2$, which implies $\alpha = \beta/2 = 1/3$.

Optimal quota of tourists (t^0) In Santorini (Greece), cruise ship visitors have been capped at 8000 per day due to overtourism after years in which the island was receiving up to 18000. Hence, t^0 should be around $8000/18000 \approx 0.44$. Machu Picchu (Peru) reached a peak of 4488 visitors per day. UNESCO (MINCETUR, 2018) believes the limit should be 2500 to preserve the ruins, so t^0 should be around $2500/4488 \approx 0.56$. Hence, we assume an intermediate value, $t^0 = \frac{1}{2}$.

Hence:

$$\begin{aligned} u_3 &= 1 - \frac{1}{3}\left|\tau - \frac{1}{2}\right| - \frac{2}{3}(1-p)m - \frac{2}{3}p + \varepsilon s \\ &= 1 - \frac{1}{6}|2\tau - 1| - \frac{2}{3}((1-p)m + p) + \varepsilon s. \end{aligned}$$

We normalize these utilities so that, apart from the marginal transfers, the maximum a player can get is 1 and the minimun is 0. Hence, the new utility functions are the following:

$$\begin{aligned} \overline{u}_1 &= u_1 = (1-p)m + \epsilon r \\ \overline{u}_2 &= u_2 = \tau - r - s \\ \overline{u}_3 &= \frac{6}{5}u_3 - \frac{1}{5} = 1 - \frac{1}{5}|2\tau - 1| - \frac{4}{5}\left((1-p)m + p\right) + \frac{6}{5}\varepsilon s. \end{aligned}$$

This normalization does not affect our results because the ordinality of SS. The other solutions (Harsanyi value, Shapley NTU value and consistent value) are also independent of utility changes using an affine map.

We can then compute the worth of each coalition:

- $V(\{1\}) = (-\infty, d_1]$ where

$$d_1 = \max_m \left(\min_{t,p,r} \overline{u}_1 \right) = 0.$$

- $V(\{2\}) = (-\infty, d_2]$ where

$$d_2 = \max_{t,r,s} \left(\min_{m,p} \overline{u}_2 \right) = 0.$$

- $V(\{3\}) = (-\infty, d_3]$ where

$$d_3 = \max_p \left(\min_{m,t,s} \overline{u}_3 \right) = 0.$$

- $V(\{1,2\})$ is given by the marginal transfers from 2 to 1, since $p = 1$ removes any benefit of cooperation:

$$\begin{aligned} V(\{1,2\}) &= \{(\epsilon r, -r) : r \geq 0\} - \mathbb{R}_+^{\{1,2\}} \\ &= \{(x_1, x_2) : x_1 + \epsilon x_2 \leq 0, x_2 \leq 0\}. \end{aligned}$$

- $V(\{1,3\})$ is given by $p = 0$ and the agreement level of m, whereas $t = r = s = 0$ harms players 1 and 3 as much as possible:

$$\begin{aligned} V(\{1,3\}) &= \left\{ \left(m, \frac{4}{5}(1-m) \right) : m \in [0,1] \right\} - \mathbb{R}_+^{\{1,3\}} \\ &= \left\{ (x_1, x_3) : x_1 + \frac{5}{4} x_3 \leq 1, x_1 \leq 1, x_3 \leq \frac{4}{5} \right\}. \end{aligned}$$

- $V(\{2,3\})$ is given by the marginal transfers from 2 to 3, since $m = 1$ removes any benefit of cooperation:

$$\begin{aligned} V(\{2,3\}) &= \{(-s, \varepsilon s) : s \geq 0\} - \mathbb{R}_+^{\{2,3\}} \\ &= \{(x_2, x_3) : x_2 \leq 0, \varepsilon x_2 + x_3 \leq 0\}. \end{aligned}$$

- $V(N)$ is given by $p = 0$, the agreement levels of $m \in [0,1]$ and $t \in [.5, 1]$, and the marginal transfers from agent 2:

$$\begin{aligned} V(N) &= \{(\overline{u}_1, \overline{u}_2, \overline{u}_3), : m \in [0,1], t \in [.5,1], r, s \geq 0\} - \mathbb{R}_+^N \\ &= \{(x_1, x_2, x_3) : x_1 + x_2 \leq 1, 4x_1 + 2x_2 + 5x_3 \leq 6, 4x_1 + 6\varepsilon x_2 + 5x_3 \leq 5 + 3\varepsilon\} \\ &\quad \cap \{(x_1, x_2, x_3) : 6(1+\varepsilon) x_1 + 6\varepsilon x_2 + 5x_3 \leq 6(1+\varepsilon)\} \\ &\quad \cap \{(x_1, x_2, x_3) : x_2 \leq 1, 2x_2 + 5x_3 \leq 6, 6\varepsilon x_2 + 5x_3 \leq 5 + 3\varepsilon\} \\ &\quad \cap \{(x_1, x_2, x_3) : 6\varepsilon x_1 + 6\epsilon\varepsilon x_2 + 5\varepsilon x_3 \leq 6\varepsilon\} \end{aligned}$$

which is depicted (together with $V(\{1,3\})$ in Figure 2.

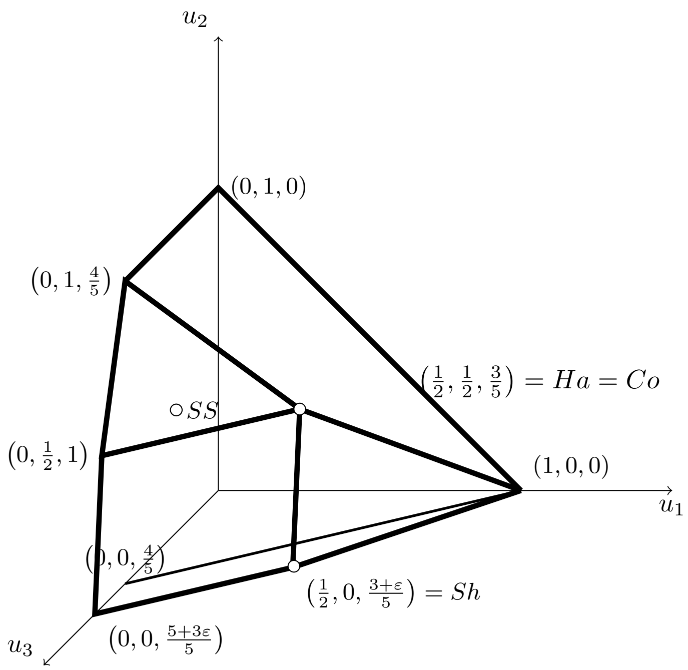

Figure 2: Pareto frontiers of $V(N)$ (thicker lines) and $V(\{1,3\})$ (thinner line) in the positive quadrant, together with the ordinal Shapley-Shubik value (SS), the Harsanyi value (Ha), the Shapley NTU value (Sh) and the consistent value (Co) for this game.

We compute the ordinal Shapley-Shubik value and compare it to the Harsanyi value, the Shapley NTU value, and the consistent value.

Ordinal Shapley-Shubik value It is straightforward to check that $x^V = \left(\frac{4\epsilon}{5\varepsilon+4\epsilon}, \frac{-4}{5\varepsilon+4\epsilon}, \frac{4\varepsilon}{5\varepsilon+4\epsilon}\right)$ and hence $x^* = \left(\frac{4\epsilon}{5\varepsilon+4\epsilon}, 0, \frac{4\varepsilon}{5\varepsilon+4\epsilon}\right)$. Recall that ϵ and ε are the marginal benefits that the tourism industry can provide respectively to the mining industry and to the local communities. As compared with the benefit obtained by ore extraction, we can assume ϵ is negligible with respect to ε, and hence x^* is approximately: $x^* \approx \left(0, 0, \frac{4}{5}\right)$ from where we deduce

$$SS(V) \approx \left(\frac{1}{12}, \frac{2}{3}, \frac{13}{15}\right)$$

attainable by setting $m = \frac{1}{12}$, $t = \frac{2}{3}$, and $p = r = s = 0$, i.e., a relatively low mining activity, a slight over tourism, and no protests nor marginal transfers.

Harsanyi value There exists a Harsanyi value obtained by setting $\lambda = \left(\frac{30}{61}, \frac{6}{61}, \frac{25}{61}\right)$, $\xi^{\{1,3\}} = \frac{12}{61}$, $\xi^N = \frac{3}{61}$, and $\xi^S = 0$ otherwise, which results in

$$Ha(V) = \left(\frac{1}{2}, \frac{1}{2}, \frac{3}{5}\right)$$

attainable by setting $m = \frac{1}{2}$, $t \in \left[\frac{1}{2}, 1\right]$ (hence, $\tau = \frac{1}{2}$), and $p = r = s = 0$, i.e., half mining activity, no over tourism, no protests, and no marginal transfers.

Shapley NTU value There exists a Shapley NTU value obtained by setting $\lambda = \left(\frac{6+5\varepsilon}{11+10\varepsilon}, \frac{5\varepsilon}{11+10\varepsilon}, \frac{5}{11+10\varepsilon}\right)$, $v^\lambda(\{1,3\}) = v^\lambda(N) = \frac{6+5\varepsilon}{11+10\varepsilon}$, and $v^\lambda(S) = 0$ otherwise, which results in $Sh(v^\lambda) = \left(\frac{6+5\varepsilon}{22+20\varepsilon}, 0, \frac{6+5\varepsilon}{22+20\varepsilon}\right)$ and

$$Sh(V) = \left(\frac{6+5\varepsilon}{(22+20\varepsilon)\lambda_1}, \frac{0}{\lambda_2}, \frac{6+5\varepsilon}{(22+20\varepsilon)\lambda_3}\right) = \left(\frac{1}{2}, 0, \frac{6+5\varepsilon}{10}\right)$$

attainable by setting $m = \frac{1}{2}$, $t \in \left[\frac{1}{2}, 1\right]$ (hence, $\tau = \frac{1}{2}$), $s = \frac{1}{2}$, and $p = r = 0$, i.e., half mining activity, no over tourism, no protests, and the tourist industry (inefficiently) transfers all its benefit of cooperation to the local communities. Notice that this payoff allocation converges to $(.5, 0, .6)$, weakly below $(.5, .5, .6)$, as ε approaches zero.

Consistent value There exists a unique consistent value obtained by setting $\lambda^{\{1,3\}} = \left(\frac{4}{9}, \frac{5}{9}\right)$, $\lambda^N = \left(\frac{7}{13}, \frac{1}{13}, \frac{5}{13}\right)$, $v^{\lambda^{\{1,3\}}}(\{1,3\}) = \frac{4}{9}$, $v^{\lambda^N}(N) = \frac{7}{13}$, and $v^{\lambda^S}(S) = 0$ otherwise, which results in

$$Co(V) = \frac{1}{6} \sum_{\pi \in \Pi^N} Co^{\pi}(V) = \left(\frac{1}{2}, \frac{1}{2}, \frac{3}{5}\right)$$

attainable, as the Harsanyi value, by setting $m = \frac{1}{2}$, $t \in \left[\frac{1}{2}, 1\right]$ (hence, $\tau = \frac{1}{2}$), and $p = r = s = 0$, i.e., half mining activity, no over tourism, no protests, and no marginal transfers.

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